Project C Advance of the Perihelion of Mercury

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- *How much do Newton and Einstein disagree about the orbit of Mercury?*
- *How do theory and experiment decide between them?*
- *Why Mercury?*

Project C

Advance of the Perihelion of Mercury

This discovery was, I believe, by far the strongest emotional experience in Einstein's scientific life, perhaps in all his life. Nature had spoken to him. He had to be right. "For a few days, I was beside myself with joyous excitement." Later, he told Fokker that his discovery had given him palpitations of the heart. What he told de Haas is even more profoundly significant: when he saw that his calculations agreed with the unexplained astronomical observations, he had the feeling that something actually snapped in him.

—Abraham Pais

1 Joyous Excitement

What discovery sent Einstein into "joyous excitement" in November of 1914? It was the calculation showing that his brand new (actually not quite completed) theory of general relativity gave the correct value for one detail of the orbit of the planet Mercury that had previously been unexplained.

Mercury circulates around Sun in a not quite circular orbit: The planet oscillates in and out radially while it circles tangentially. The result is an elliptic orbit. Newton tells us that if we consider only the interaction between planet and Sun, then the time for one circular orbit is *exactly* the same as one in-and-out radial oscillation. Therefore the orbital point closest to Sun, the so-called **perihelion**, stays in the same place; the elliptical orbit does not shift around with each revolution—according to Newton. In this project you will begin by verifying this nonrelativistic result. Why bother calculating something that does not change? Because observation shows that Mercury's orbit does, in fact, change. The innermost point, the perihelion, moves around the Sun a *little*; it *advances* with each orbit (Figure 1). The long (major) axis of the ellipse rotates at the tiny rate of 574 seconds of arc (0.159 degree) *per century.* (One degree equals 3600 seconds of arc.) Newtonian mechanics accounts for 531 seconds of this advance by computing the perturbing influence of the other planets. But a stubborn 43 seconds of arc (0.0119 degree) per century (called a **residual**) remains after all these effects are accounted for. This discrepancy (though not its modern value) was computed from observations by LeVerrier as early as 1859.

Simon Newcomb

Simon Newcomb
Born March 12, 1835, Wallace, Nova Scotia
Died July 11, 1909, Washington, D.C.
(Photo courtesy of Yerkes Observatory)

From 1901 until 1959 and even later, the tables of locations of the planets (so-called **ephmerides**) used by most astronomers were those compiled by Simon Newcomb and his collaborator, George W. Hill. By the age of five Newcomb was spending several hours a day making calculations and before the age of seven was extracting cube roots by hand. He had little formal education but avidly explored many technical fields in the libraries of Washington, D. C. He discovered the *American Ephemeris and Nautical Almanac*, of which he said, "Its preparation seemed to me to embody the highest intellectual power to which man had ever attained."

Newcomb became a "computer" (someone who computes) in the American Nautical Almanac Office and, by stages, rose to become its head. The greater part of the rest of his life was spent calculating the motions of bodies in the solar system from the best existing data. Newcomb collaborated with Q. M. W. Downing to inaugurate a worldwide system of astronomical constants, which was adopted by many countries in 1896 and officially by all countries in 1950.

The advance of the perihelion of Mercury computed by Einstein in 1914 would have been compared to entries in the tables of Simon Newcomb.

The advance of the perihelion of Mercury is sometimes called the **precession of the perihelion.**

Newtonian mechanics says that there should be *no residual* advance of the perihelion of Mercury's orbit and so cannot account for the 43 seconds of arc per century which, though tiny, is nevertheless too large to be ignored or blamed on observational error. But Einstein's general relativity hit it on the button. Result: joyous excitement!

In this project we review Newton's incorrect prediction and then carry out a general-relativistic approximate calculation of the advance of the perihelion of Mercury adapted from that of Robert M. Wald (*General Relativity*, University of Chicago Press, 1984, pages 142–143). This approximation describes the angular motion of the planet as if it were in a nearly circular orbit. From this assumption we calculate the time for one orbit. The approximation also describes the small inward and outward radial motion of the planet as if it were a harmonic oscillator moving back and forth radially about the minimum in a potential well (Figure 2). We calculate the time for one round-trip radial oscillation. These two times are equal, according to Newton, if one considers only the planet-Sun interaction. In that case the planet goes around once in the same time that it oscillates radially inward and back out again. The result is an elliptical orbit that closes on itself, so the planet repeats its elliptic path forever. In contrast, these two times—the angular and the radial—are *not quite* equal according to the Einstein approximation. The radial oscillation takes place more slowly. From the difference we reckon the approximate rate of advance of Mercury's perihelion around Sun.

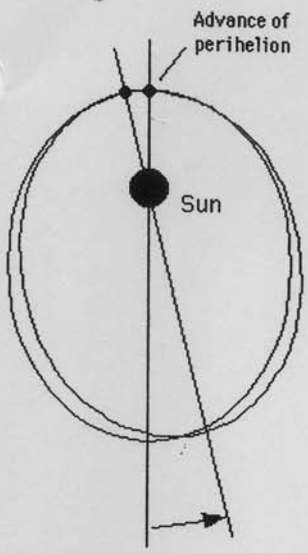

Figure 1 *Exaggerated view of the change in orientation of Mercury's orbit during one century.*

2 Linear Harmonic Oscillator

Why should the satellite oscillate in and out radially? Look at the effective potential for Newtonian motion, the heavy line in Figure 2. This heavy line has a minimum, the location at which a particle can rest and ride around at constant *r*, executing a circular orbit. But it can also oscillate radially in and out, as shown by the two-headed arrow.

How long will it take for one in-and-out oscillation? That depends on the shape of the effective potential curve near the minimum shown in Figure 2. If the amplitude of the oscillation is small, then the important part of the curve is very close to this minimum, and we can use a well-known mathematical theorem: If a continuous, smooth curve has a minimum, then near that minimum the curve can be approximated by a parabola with its vertex at the minimum point. Such a parabola is shown superimposed on the effective potential curve of Figure 2. From the diagram it is apparent that the parabola is a good approximation of the potential near that minimum. In fact Mercury's orbit swings from a minimum radius (the perihelion) of 46.04 million kilometers to a maximum radius (the so-called **aphelion**) of 69.86 million kilometers.

From introductory physics we know how a particle moves in a parabolic potential. The motion is called **harmonic oscillation** and follows a formula of the kind

$$x = A \sin \omega t \qquad [1]$$

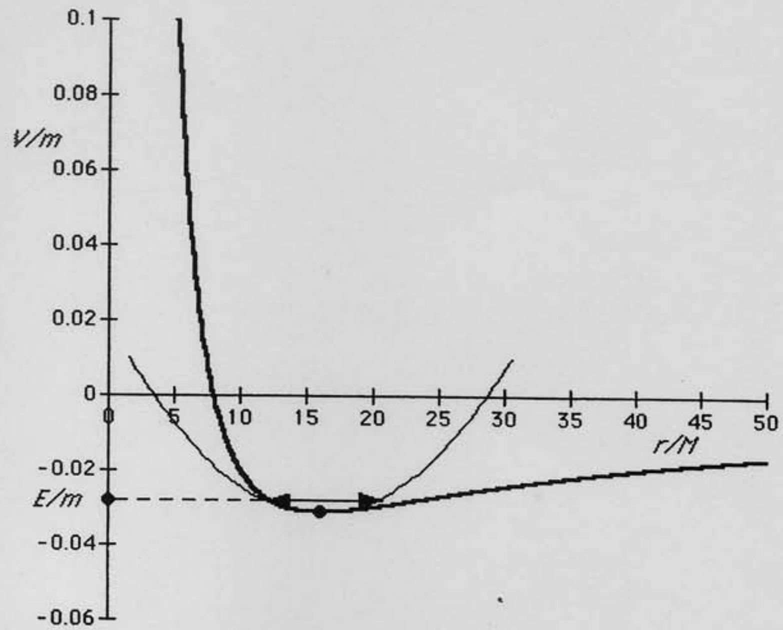

Figure 2 *Computer plot: The Newtonian effective potential (thick curve), copied from Figure 5, page 4-12, on which is superimposed the parabolic potential of the simple harmonic oscillator (thin curve). The two curves conform to one another only near the minimum of the effective potential. We use a similar set of curves to approximate the radial oscillation of Mercury in its orbit as an harmonic oscillation of small amplitude.*

Here A is the amplitude of the oscillation and ω (Greek lower-case omega) tells us how rapidly the oscillation occurs. The potential energy per unit mass V/m of a particle oscillating in a parabolic potential is given by the formula

$$V/m = \frac{1}{2}\omega^2 x^2 \qquad [2]$$

From equation [2] we can find an expression for ω by taking the second derivative of both sides with respect to the displacement x:

$$\frac{d^2(V/m)}{dx^2} = \omega^2 \qquad [3]$$

In general, if we have the expression for the potential, we can find the rate ω of harmonic oscillation around a minimum by taking the second derivative of the curve and evaluating it at that minimum where $d(V/m)/dx = 0$.

3 Radial Harmonic Oscillation of Mercury: Newton

The trouble with the in-and-out radial oscillation of Mercury is that it does not take place around $x = 0$ but around the average radius r_o of its orbit.

What is the value of r_0? It is the radius for which the effective potential has a minimum. For Newtonian orbits the radial motion is given by equations [27], page 4-11, and [29], page 4-12:

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = \frac{E}{m} - \left[-\frac{M}{r} + \frac{(L/m)^2}{2r^2}\right] = \frac{E}{m} - \frac{V(r)}{m} \qquad \text{[4. Newton]}$$

From this equation we define the effective potential (equation [28] on page 4-12):

$$\frac{V(r)}{m} = -\frac{M}{r} + \frac{(L/m)^2}{2r^2} \qquad \text{[5. Newton]}$$

QUERY 1 **Finding the potential minimum.** Take the derivative with respect to r of the potential per unit mass, V/m given in equation [5]. Set this first derivative aside for use in Query 2. As a separate calculation, equate this derivative to zero in order to determine the radius r_o at the effective potential minimum. Use the result to write down an expression for the unknown quantity $(L/m)^2$ in terms of the known quantities M and r_o.

QUERY 2 **Oscillation rate ω_r for radial motion.** We want to use equation [3] to find the rate of radial oscillation. Accordingly, continue by taking a second derivative of V/m in equation [5] with respect to r. Set $r = r_o$ in the resulting expression and substitute your value for $(L/m)^2$ from Query 1. Use equation [3] to find an expression for the rate at which Mercury oscillates in and out radially—according to Newton!

4 Angular Velocity of Mercury in Its Orbit: Newton

We want to compare the rate ω_r of in-and-out radial motion of Mercury with its rate ω_ϕ of round-and-round tangential motion. Use the Newtonian definition of angular momentum, with increment dt of Newtonian universal time, similar to equation [2], page 4-3:

$$L/m = r^2\frac{d\phi}{dt} = r^2\omega_\phi \qquad \text{[6. Newton]}$$

We want to find the value for the angular velocity $\omega_\phi = d\phi/dt$ of Mercury along its almost circular orbit.

QUERY 3 **Angular velocity of Mercury in orbit.** Into equation [6] substitute your value for L/m from Query 1 and set $r = r_o$. Find an expression for $d\phi/dt$ in terms of M and r_o.

QUERY 4 **Comparing radial oscillation rate with orbital angular velocity.** Compare your value of angular velocity ω_ϕ from Query 3 with your value for radial oscillation rate ω_r from Query 2. State your conclusion about the advance of the perihelion of Mercury's orbit around Sun (when only the Sun-Mercury interaction is considered), according to Newton.

5 Effective Potential: Einstein

Now we repeat the analysis for the general relativistic case, using the Newtonian analysis as our model. Equation [30], page 4-15 gives a measure of the radial motion of the orbiting planet. Multiply through by 1/2 to obtain an equation similar to equation [4] above for the Newtonian case:

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 = \frac{1}{2}\left(\frac{E}{m}\right)^2 - \frac{1}{2}\left(1-\frac{2M}{r}\right)\left[1+\frac{(L/m)^2}{r^2}\right] = \frac{1}{2}\left(\frac{E}{m}\right)^2 - \frac{U}{m} \qquad [7]$$

Equations [4] and [7] are of similar form, and we use this similarity to make a harmonic analysis of the radial motion of Mercury in orbit in general relativity similar to the Newtonian analysis of Sections 3 and 4. Begin by assigning the name *effective potential* and the symbol *U/m* to the term subtracted from the squared energy in [7], as indicated on the right end of the equation.

Before proceeding further, note first that the time in equation [7] is the proper time τ, the wristwatch time of the satellite, not Newton's universal time *t*. This different time standard is not necessarily fatal, since in Newtonian mechanics there is only one universal time, and we have not yet had to decide which relativistic time should replace it. You will show that for Mercury the choice of which time to use (wristwatch time, bookkeeper far-away time, or even shell time at the radius of the orbit) makes a negligible difference in our predictions about the rate of advance of the perihelion.

Second, note that the relativistic expression $(1/2)(E/m)^2$ in equation [7] stands in the place of the Newtonian expression (E/m) in equation [4]. Do we dare replace an energy with a squared energy? Both represent a constant of the motion and, strange as it may seem, the difference does not affect our analysis. Evidence that we are on the right track follows from multiplying out the second term of the middle equality in equation [7]. We have assigned the symbol *U/m* to this second term.

$$\begin{aligned}\frac{U}{m} &= \frac{1}{2}\left(1-\frac{2M}{r}\right)\left[1+\frac{(L/m)^2}{r^2}\right] \\ &= \frac{1}{2} - \frac{M}{r} + \frac{(L/m)^2}{2r^2} - \frac{M(L/m)^2}{r^3}\end{aligned} \qquad [8]$$

On the right side of the second line are the two effective potential terms that made up the Newtonian expression [5]. In addition, the first term (1/2) assures that far from the center of attraction the radial speed in [7] will have the correct value. For example, let the total energy equal the rest energy ($E/m = 1$). Then for large r, the radial speed $dr/d\tau$ (equation [7]) goes to zero, as it must in this case. The potential U/m is plotted in Figure 3.

The final term on the right of the second line of [8] describes an attractive potential arising from general relativity. This causes the slight deviation of the orbit of Mercury from that predicted by Newton. Because of the r^3 in the denominator, near a black hole this negative term overwhelms all others at small radii, leading to the downward plunge in the effective potential at the left side of Figure 3.

In summary, the forms of equations [7] and [8] allow us to use the tools of Newtonian mechanics to analyze the radial component of the satellite's motion predicted by general relativity, provided that we are satisfied with the wristwatch time of the satellite and with an "energy term" equal to $(1/2)(E/m)^2$. Of course, we are trying to solve a relativistic problem. Nevertheless, because of its form we can use the Newtonian manipulation to carry out a general relativistic calculation.

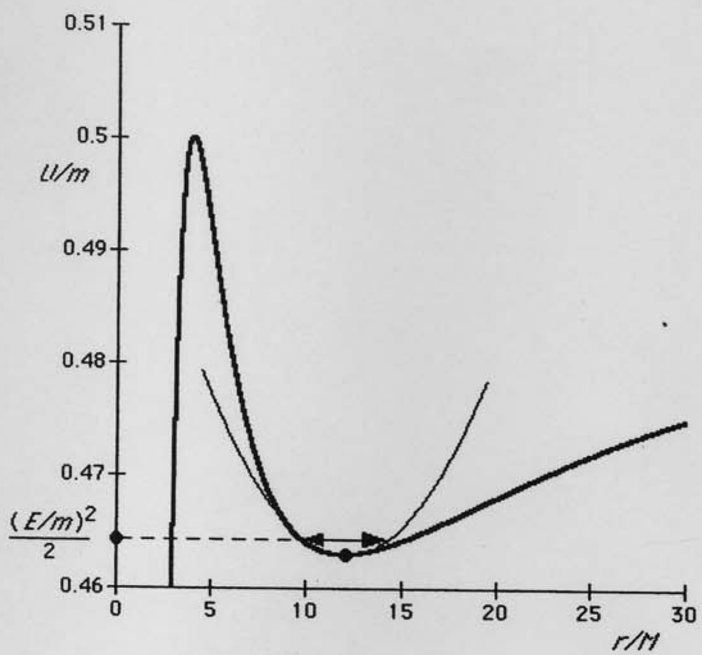

Figure 3 *Computer plot: Approximation of the general-relativistic effective potential U/m (heavy curve) at the minimum with a parabola (light curve) in order to analyze the radial excursion (double-headed arrow) as simple harmonic motion. The heavy effective potential curve is for a black hole, not for Sun, whose effective potential would be indistinguishable from the Newtonian function on the scale of this diagram.*

6 Radial Harmonic Oscillation of Mercury: Einstein

Now analyze the radial oscillation of Mercury according to Einstein.

QUERY 5 **Finding the potential minimum.** Take the derivative of the effective potential [8] with respect to *r*. Set this first derivative aside for use in Query 6. As a separate calculation, equate this derivative to zero, set $r = r_o$, and solve the resulting equation for the unknown quantity $(L/m)^2$ in terms of the known quantities M and r_o.

QUERY 6 **Radial oscillation rate.** We want to use equation [3] to find the rate of oscillation in the radial direction. Accordingly, continue to the second derivative of U/m from equation [8]. Set $r = r_o$ in the result and substitute the expression for $(L/m)^2$ from Query 5 to obtain

$$\left.\frac{d^2(U/m)}{dr^2}\right|_{r=r_0} = \omega_r^2 = \frac{M(r_o - 6M)}{r_o^3(r_o - 3M)} \qquad [9]$$

QUERY 7 **Newtonian limit of radial oscillation.** The radius of Mercury's orbit around Sun has the value $r_o = 5.80 \times 10^{10}$ meters. Compare this radius with the value M for the mass of Sun in geometric units. If one of these can be neglected in equation [9] compared with the other, demonstrate that the resulting value of ω_r is the same as your Newtonian expression derived in Query 2.

7 Angular Velocity in Orbit: Einstein

We want to compare the rate of in-and-out radial oscillation of Mercury with the angular rate at which Mercury moves tangentially in its orbit. The rate of change of azimuth ϕ springs from the definition of angular momentum, equation [2], page 4-3:

$$\frac{L}{m} = r^2 \frac{d\phi}{d\tau} \qquad [10]$$

Note that the time here, too, is the wristwatch (proper) time τ of the satellite.

QUERY 8 **Angular velocity.** Square both sides of equation [10] and use your result from Query 5 to eliminate $(L/m)^2$ from the resulting equation. Show that the result can be written

$$\omega_\phi^2 \equiv \left(\frac{d\phi}{d\tau}\right)^2 = \frac{M}{r_o^2(r_o - 3M)} \qquad [11]$$

According to the relativistic prediction, does the round-and-round tangential motion of Mercury take place in step with the in-and-out radial oscillation, as it does in the Newtonian analysis?

QUERY 9 **Newtonian limit of angular velocity.** Make the same kind of approximation as in Query 7 and demonstrate that the resulting value of ω_ϕ is the same as your Newtonian expression derived in Query 3.

8 Predicting Advance of the Perihelion

The advance of the perihelion of Mercury springs from the difference between the frequency at which the planet sweeps around in its orbit and the frequency at which it oscillates in and out radially. In the Newtonian analysis these two frequencies are equal if one considers only the interaction between planet and Sun. But Einstein's theory shows that these two frequencies are *not quite* equal, so Mercury reaches its maximum (or minimum) radius at a slightly different angular position in each orbit. This results in the advance of the perihelion. The rate of advance is the difference between the orbital angular frequency ω_ϕ and the radial angular frequency ω_r.

QUERY 10 **Difference in squared oscillation rates.** From equations [11] and [9] construct and simplify an expression for the difference of squares $\omega_\phi^2 - \omega_r^2$ in terms of M, r_o, and ω_ϕ plus numerical constants.

QUERY 11 **Difference in oscillation rates.** The two angular rates ω_ϕ and ω_r are *almost* identical in value, even in the Einstein analysis. Therefore write the result of Query 10 in the following form:

$$\omega_\phi^2 - \omega_r^2 = (\omega_\phi + \omega_r)(\omega_\phi - \omega_r) \approx 2\omega_\phi(\omega_\phi - \omega_r) \qquad [12]$$

Use outcomes of earlier queries to show that this approximation can be written

$$\omega_\phi - \omega_r \approx \frac{3M}{r_o}\omega_\phi \qquad [13]$$

Equation [13] gives us the difference in angular rate between the tangential motion and the radial oscillation. From this rate difference we can calculate the rate of advance of the perihelion of Mercury.

All of the ω-expressions are of the form $d(\text{angle})/d\tau$ or $d(\text{phase angle})/d\tau$. Since $d\tau$ is in the denominator everywhere, it can be canceled out and the angle increments added to give angles. The resulting adaptation of equation [13] has the following form:

$$\begin{pmatrix}\text{predicted}\\ \text{angle of}\\ \text{advance}\end{pmatrix} = \begin{pmatrix}\text{total angle}\\ \text{covered in}\\ \text{orbital motion}\end{pmatrix} - \begin{pmatrix}\text{total phase angle}\\ \text{covered in}\\ \text{radial motion}\end{pmatrix}$$

$$= \frac{3M}{r_o}\begin{pmatrix}\text{total angle}\\ \text{covered in}\\ \text{orbital motion}\end{pmatrix} \qquad [14]$$

Moreover, we can use any measure of angle we wish—degrees or radians or seconds of arc—as long as we are consistent. Numerical prediction based on this equation must be compared with results of observation.

9 Comparison with Observation

QUERY 12 **Mercury's orbital period.** The period of Mercury's orbit is 7.602×10^6 seconds and that of Earth is 3.157×10^7 seconds. What is the value of Mercury's period in Earth-years?

QUERY 13 **Mercury's revolution in one century**. How many revolutions around Sun does Mercury make in one century (100 Earth-years)? How many degrees of angle are traced out by Mercury in one century?

QUERY 14 **Correction factor.** The mass M of Sun is 1.477×10^3 meters and the radius r_o of Mercury's orbit is 5.80×10^{10} meters. Calculate the value of the correction factor $3M/r_o$ in equation [14].

QUERY 15 **Advance angle per century in degrees.** Using equation [14], multiply your answers from Queries 13 and 14 to obtain a prediction of the advance of the perihelion of Mercury's orbit per century in degrees.

QUERY 16 **Advance angle per century in seconds of arc.** There are 60 minutes of arc per degree and 60 seconds of arc per minute of arc. Multiply your result from Query 15 by $60 \times 60 = 3600$ to obtain your prediction of the advance of the perihelion of Mercury's orbit per century in seconds of arc.

A more careful analysis predicts a value of 42.98 seconds of arc (0.0119 degrees) per century (see Table 1). The observed rate of advance of the perihelion is in perfect agreement with this value: 42.98 ± 0.1 seconds of arc per century. (See references.) How close was your prediction?

10 Advance of the Perihelia of the Inner Planets

Do the *perihelia* (plural of *perihelion*) of other planets in the solar system also advance as described by general relativity? Yes, but these planets are farther from Sun, so the predicted advance is less than that of Mercury. In this section we compare our estimated advance of the parahelia of the inner planets Mercury, Venus, Earth, and Mars with results of an accurate calculation.

The Jet Propulsion Laboratory (JPL) in Pasadena, California, supports an active effort to improve our knowledge of the positions and velocities of the major bodies in the solar system. For the major planets and the moon, JPL maintains a database and set of computer programs known as the Solar System Data Processing System (SSDPS). The input database contains the observational data measurements for current locations of the planets. Working together, more than 100 interrelated computer programs use these data and the relativistic laws of motion to compute locations of planets at times in the past and future. The equations of motion take into account not only the gravitational interaction between each planet and Sun but also interactions among all planets, Earth's moon, and 300 of the most massive asteroids, as well as interactions between Earth and Moon due to nonsphericity and tidal effects.

To help us with our project on perihelion advance, Myles Standish, Principal Member of the Technical Staff at JPL, kindly used the numerical integration program of the SSDPS to calculate orbits of the four inner planets over four centuries, from A.D. 1800 to A.D. 2200. In an overnight run he carried out this calculation twice, once with the full program including relativistic effects and a second time "with relativity turned off." Standish "turned off relativity" by setting the speed of light to 10^{10} times its measured value, effectively making light speed infinite. (By combining equation [5], page 2-14, with equation [10], page 2-19, we can show that the Schwarzschild curvature factor in conventional units is written $(1 - 2GM_{kg}/rc^2)$; the value of this expression approaches unity for a large value of c.) For each of the two runs, the perihelia of the four inner planets were computed for a series of points in time covering the four centuries. The results from the nonrelativistic run were subtracted from those of the relativistic run, revealing advances of the perihelia per century accounted for only by general relativity. The second column of Table 1 shows the results, together with the estimated computational error. Later columns show additional data on these planets.

Table 1 Advance of the perihelia of the inner planets

Planet	Advance of perihelion in seconds of arc per century (JPL calculation)	Radius of orbit in AU*	Period of orbit in years
Mercury	42.980 ± 0.001	0.38710	0.24085
Venus	8.618 ± 0.041	0.72333	0.61521
Earth	3.846 ± 0.012	1.00000	1.00000
Mars	1.351 ± 0.001	1.52368	1.88089

*Astronomical Unit (AU): average radius, Earth's orbit; inside back cover.

QUERY 17 **Perihelia advance of the inner planets.** Compare the JPL-computed advances of the perihelia of Venus, Earth, and Mars with results of the approximate formula developed in this project.

11 Checking the Standard of Time

QUERY 18 **Difference between shell and wristwatch times.** Use special relativity to find the fractional difference between satellite wristwatch time τ and the time t_{shell} read on shell clocks at the same radius r_o at which Mercury moves in its orbit at the average velocity 4.8×10^4 meters/second. By what fraction could a change of time from τ to t_{shell} change the total angle covered in the orbital motion of Mercury in one century (equation [14])? Therefore by what fraction could it change the predicted angle of rotation of the major axis?

QUERY 19 **Difference between shell and far-away times.** Find the fractional difference between shell time t_{shell} at radius r_o and bookkeeper far-away time t for r_o equal to the radius of the orbit of Mercury. By what fraction could a change of time from t_{shell} to t change the total angle covered in the orbital motion of Mercury in one century (equation [14])? Therefore by what fraction will it change the predicted angle of axis rotation?

QUERY 20 **Does time standard matter?** From your results of Queries 18 and 19, say whether or not the choice of a time standard (planet proper time τ, t_{shell}, or far-away time t) would make a significant difference in the numerical prediction of the advance of the perihelion of Mercury in one century. Would your answer differ if the time were measured with clocks on Earth's surface?

12 References and Acknowledgments

Initial quote: Abraham Pais, *Subtle Is the Lord: The Science and the Life of Albert Einstein*, Oxford University Press, New York, 1982, page 253.

Observed value of the advance of the perihelion of Mercury: Irwin Shapiro in *General Relativity and Gravitation, 1989*, edited by N. Ashby et al., Cambridge University Press, New York, 1990, page 313.

Periods and orbital radii in Table 1 from two sources: Kenneth R. Lang, *Astrophysical Data, Planets and Stars*, Springer-Verlag, New York, 1991, and *Landolt-Börnstein, Numerical Data and Functional Relationships in Science and Technology, Group VI: Astronomy and Astrophysics*, Volume 34, *Astronomy and Astrophysics*, Extension and Supplement to Volume 2, Subvolume a, *Instruments, Methods, Solar System*, edited by H. H. Voigt, Springer-Verlag, New York, 1993.

Myles Standish of the Jet Propulsion Laboratory ran the programs on the inner planets presented in Section 10. He also made useful comments on the project as a whole.

Calculation of the Advancement of Mercury's Perihelion

From the General Relativity discussions: The Einstein Effective Potential is given by the expression:

$$\frac{U}{m} = \frac{1}{2} - \frac{M}{r} + \frac{\frac{L^2}{m^2}}{2r^2} - M\left(\frac{\frac{L^2}{m^2}}{r^3}\right)$$

a. **Determine the potential minimum and solve the resulting equation for L^2/m^2 in terms of M and r_0.**

b. **Determine the radial oscillation rate** $\omega_r^2 = \frac{d^2}{dr^2}\left(\frac{U}{m}\right)\Big|r = r_0$

c. **Determine the azimuthal oscillation rate** $\frac{L}{m} = r^2 \frac{d\phi}{d\tau}$

d. **Determine an expression for** $\omega_\phi^2 - \omega_r^2$ **in terms of M, r_0, and ω_ϕ plus numerical constants.**

e. **Derive the expression** $\omega_\phi - \omega_r \approx \frac{3M}{r_0}\omega_\phi$ **from the above considerations, from which we can calculate the rate of advance of the perihelion of Mercury.**

f. **Mercury's orbital period is 7.602×10^6 seconds and Earth's is 3.156×10^7 seconds. Determine Mercury's period in earth years.**

g. **How many revolutions around the sun does Mercury make in one century. Determine the number of degrees of angle traced out by Mercury's orbit in one century.**

h. **M is the sun's mass = 1.477×10^3 meters and $r_0 = 5.80 \times 10^{10}$ meters is the radius of Mercury's orbit. Obtain a prediction of the advance of Mercury's perihelion in degrees per century, and then in seconds of arc per century. Do a % difference with the accepted value of 42.98 seconds of arc per century.**